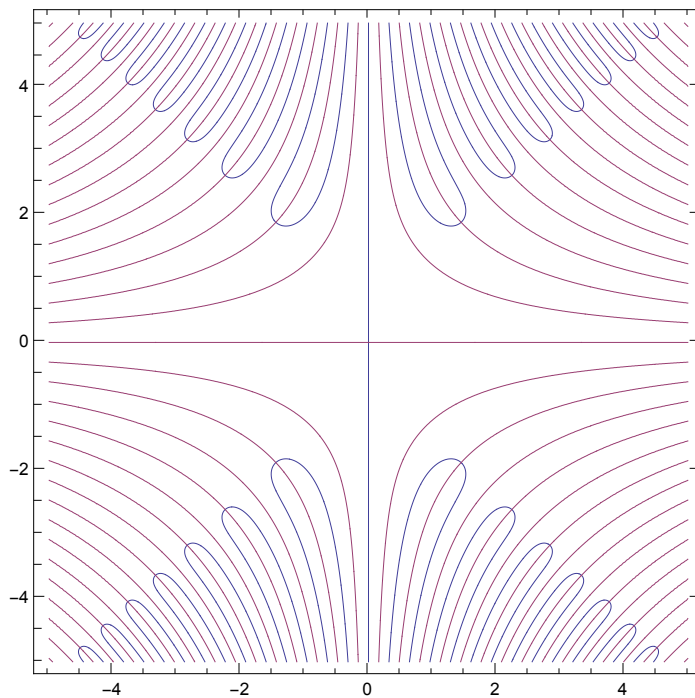


Figures for Berry's Construction

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Figure 1

```
ContourPlot[{Re[Erf[x + i y]] == 0, Im[Erf[x + i y]] == 0},  
{x, -5, 5}, {y, -5, 5}, PlotPoints -> 100]
```

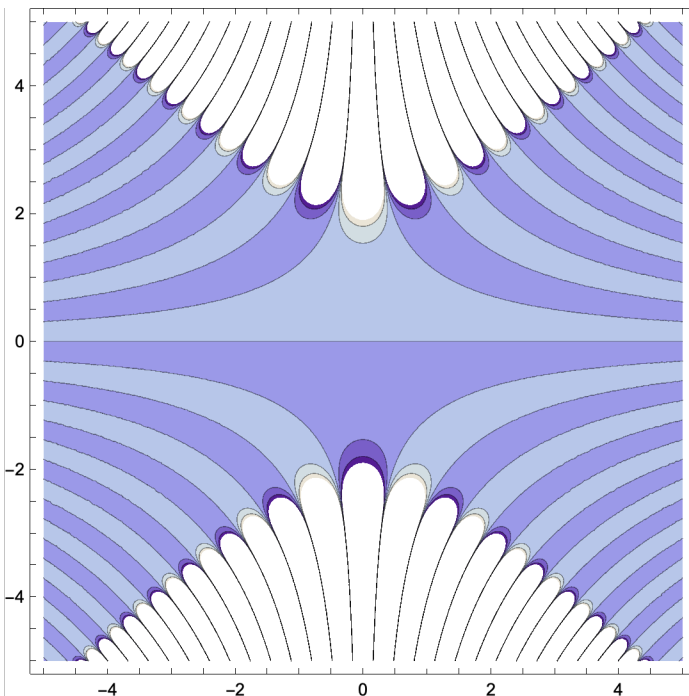
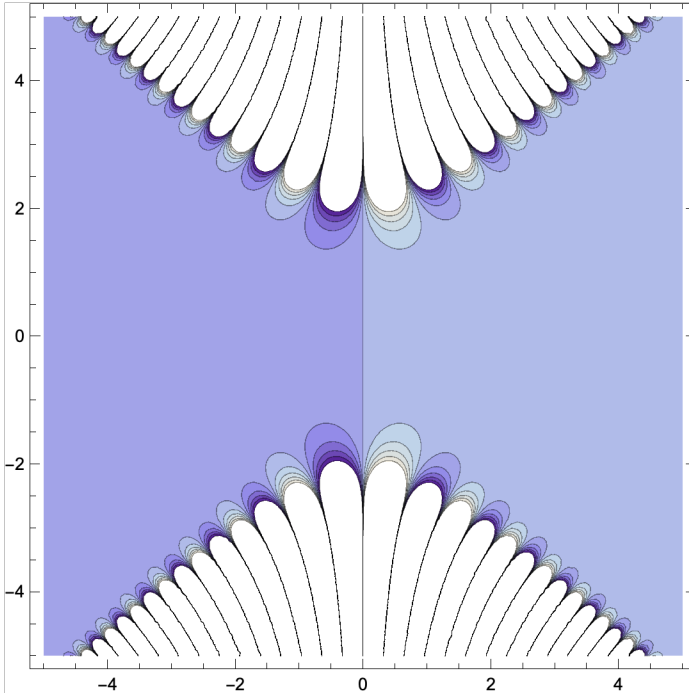


The null curves of real part (red) and imaginary part (blue) of $\text{Erf}[x + iy]$. Orthogonality at points of intersection follows from analyticity (i.e., from the Cauchy-Riemann equations).

Figures 2

```
ContourPlot[{Re[Erf[x + i y]]}, {x, -5, 5}, {y, -5, 5}]
```

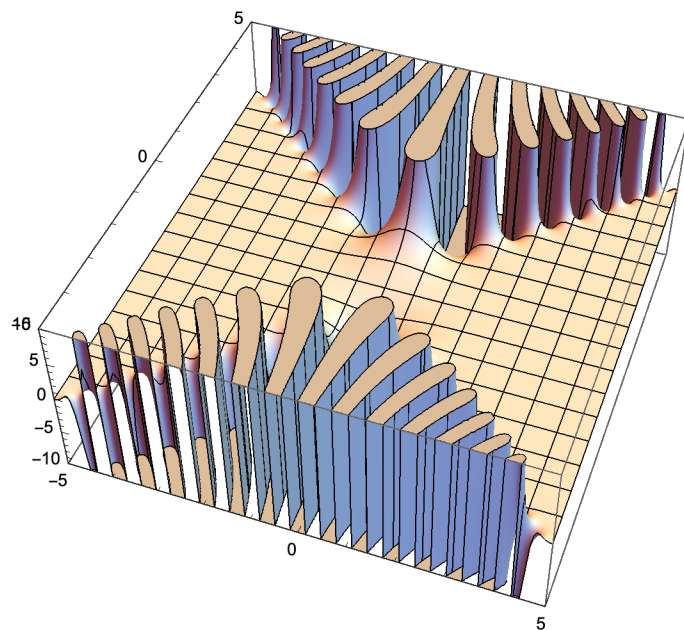
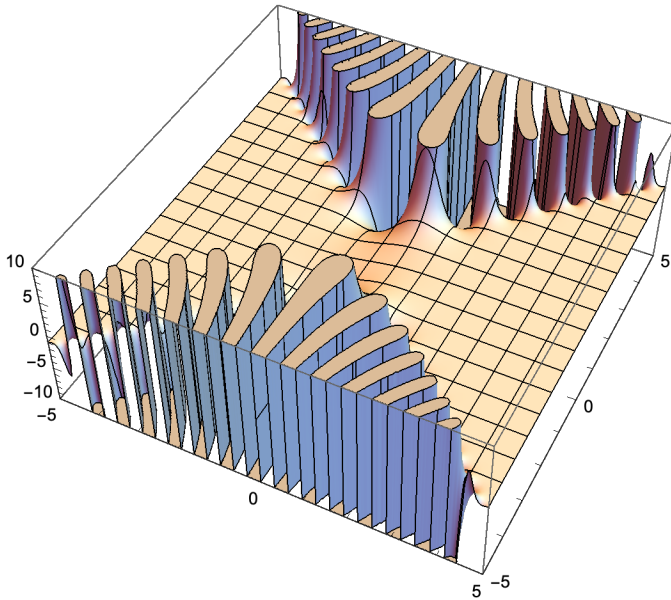
```
ContourPlot[{Im[Erf[x + i y]]}, {x, -5, 5}, {y, -5, 5}, PlotPoints -> 50]
```



Figures 3

```
Plot3D[{Re[Erf[x + i y]]}, {x, -5, 5}, {y, -5, 5},
  PlotRange → {{-5, 5}, {-5, 5}, {-10, 10}}, PlotPoints → 50]
```

```
Plot3D[{Im[Erf[x + i y]]}, {x, -5, 5}, {y, -5, 5},
  PlotRange → {{-5, 5}, {-5, 5}, {-10, 10}}, PlotPoints → 50]
```



In the flat parts of the top figure the real part of $\text{Erf}[x+iy]$ is found never to deviate by more than a few parts in a thousand from the value 1. In the bottom figure the flat portions of the imaginary part remain similarly close to the value 0. It is the flat parts of these figures that give rise to the condition $b \leq$

$$b_{\max} = 2.$$

Figure 4

```
Plot3D[{Abs[Erf[x + i y]]}, {x, -5, 5}, {y, -5, 5},  
PlotRange -> {{-5, 5}, {-5, 5}, {-10, 10}}, PlotPoints -> 50]
```

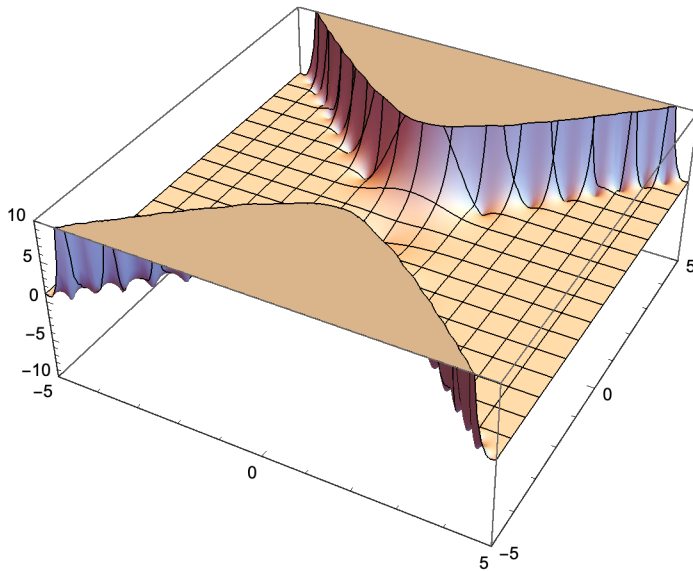
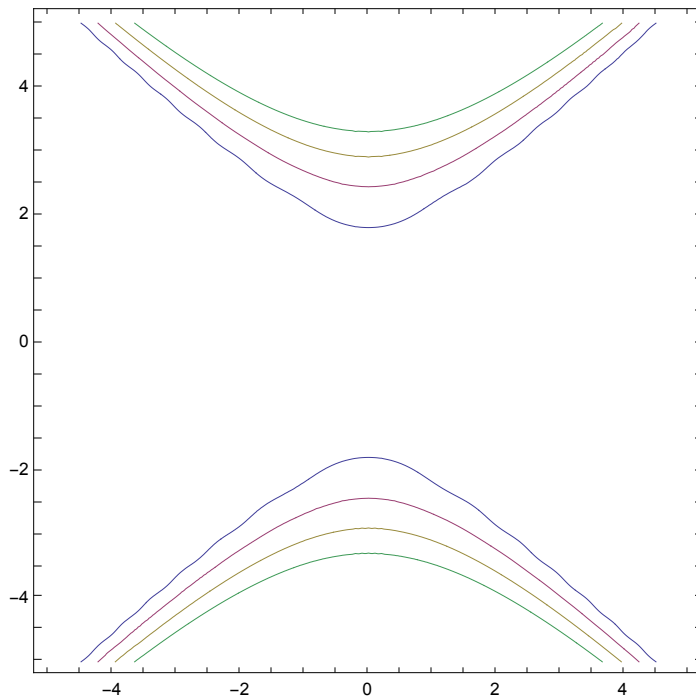


Figure 5

Yet another demonstration of how abrupt and extreme is the stratospheric ascent at the edge of the flat desert:

```
ContourPlot[{Abs[Erf[x + i y]] == 10, Abs[Erf[x + i y]] == 100, Abs[Erf[x + i y]] == 1000,
Abs[Erf[x + i y]] == 10 000}, {x, -5, 5}, {y, -5, 5}, PlotPoints -> 50]
```



Figures 6

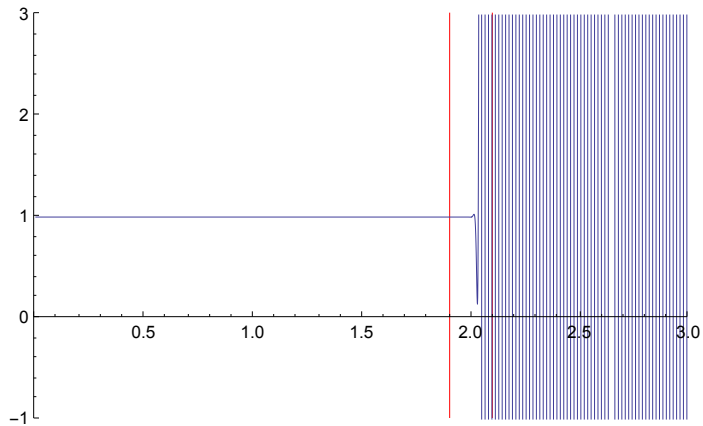
$$X[x_, v_, b_] := \frac{2 \sqrt{v} \left(\cos\left[\frac{1}{2} \operatorname{ArcTan}\left[\frac{x}{2v}\right]\right] + \left(\frac{b}{2} + \frac{x}{2v}\right) \sin\left[\frac{1}{2} \operatorname{ArcTan}\left[\frac{x}{2v}\right]\right] \right)}{\left(1 + \frac{x^2}{4v^2}\right)^{1/4}}$$

$$Y[x_, v_, b_] := \frac{2 \sqrt{v} \left(\left(\frac{b}{2} + \frac{x}{2v}\right) \cos\left[\frac{1}{2} \operatorname{ArcTan}\left[\frac{x}{2v}\right]\right] - \sin\left[\frac{1}{2} \operatorname{ArcTan}\left[\frac{x}{2v}\right]\right] \right)}{\left(1 + \frac{x^2}{4v^2}\right)^{1/4}}$$

$$Z[x_, v_, b_] := X[x, v, b] + i Y[x, v, b]$$

$$\varepsilon[x_, v_, b_] := \frac{\operatorname{Erf}[Z[x, v, b]] + \operatorname{Erf}[Z[x, v, -b]]}{2}$$

```
Plot[Re[ $\mathcal{E}[2, 50, b]$ ], {b, 0, 3}, PlotRange -> {{0, 3}, {-1, 3}},
  GridLines -> {{{1.9, Red}, {2.1, Red}}, {}]}
```



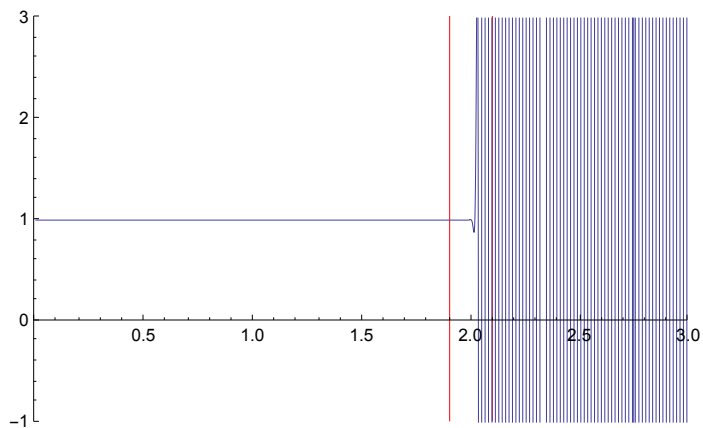
$\mathcal{E}[2, 50, 1.9]$

$\mathcal{E}[2, 50, 2.1]$

$1. - 7.1789 \times 10^{-11} i$

$1.04841 \times 10^7 + 1.68925 \times 10^7 i$

```
Plot[Re[ $\mathcal{E}[3, 50, b]$ ], {b, 0, 3}, PlotRange -> {{0, 3}, {-1, 3}},
  GridLines -> {{{1.9, Red}, {2.1, Red}}, {}]}
```



$\mathcal{E}[3, 50, 1.9]$

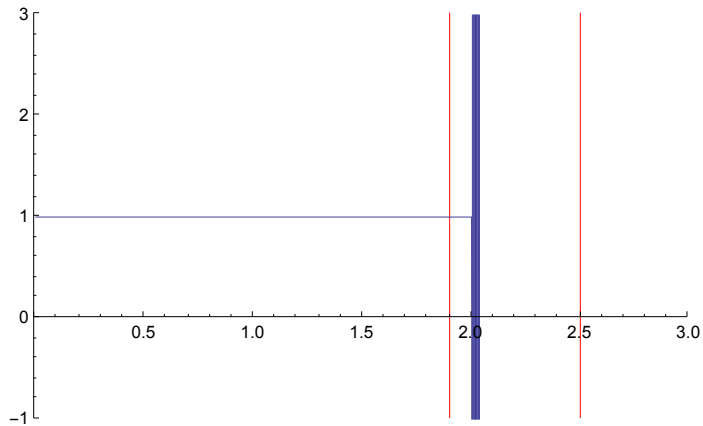
$\mathcal{E}[3, 50, 2.1]$

$1. + 6.22934 \times 10^{-11} i$

$-1.77941 \times 10^7 + 792137. i$

Large values of v can produce numbers so large that they exceed the capacity of the graphic command:

```
Plot[Re[ $\mathcal{E}[2, 5000, b]$ ], {b, 0, 3}, PlotRange → {{0, 3}, {-1, 3}},
  GridLines → {{{1.9, Red}, {2.5, Red}}, {}}
```



```
 $\mathcal{E}[2, 5000, 1.9]$ 
```

```
 $\mathcal{E}[2, 5000, 2.5]$ 
```

```
 $1. + 3.740935216178161 \times 10^{-850} i$ 
```

```
 $8.446174781711031 \times 10^{4882} - 9.14825937535729 \times 10^{4882} i$ 
```

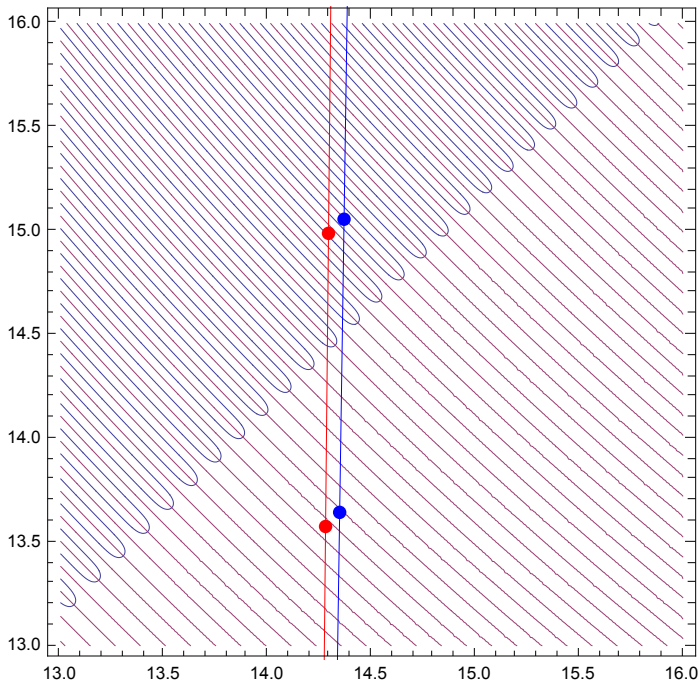
Figure 7

```
ErfFunction = ContourPlot[{Re[Erf[x + i y]] == 0, Im[Erf[x + i y]] == 0},
  {x, 13, 16}, {y, 13, 16}, PlotPoints → 50];
```

```
DataPoints = Graphics[{PointSize[Large], {Red, Point[{14.2772, 13.5744}]},
  {Red, Point[{14.2913, 14.9884}]},
  {Blue, Point[{14.3451, 13.6426}]},
  {Blue, Point[{14.3663, 15.0563}]}}];
```

```
bLine2 = ParametricPlot[{X[2, 50, b], Y[2, 50, b]},
  {b, 0, 3}, AspectRatio → Automatic, PlotStyle → {Red}];
bLine3 = ParametricPlot[{X[3, 50, b], Y[3, 50, b]}, {b, 0, 3},
  AspectRatio → Automatic, PlotStyle → {Blue}];
```

Show[ErfFunction, DataPoints, bLine2, bLine3]



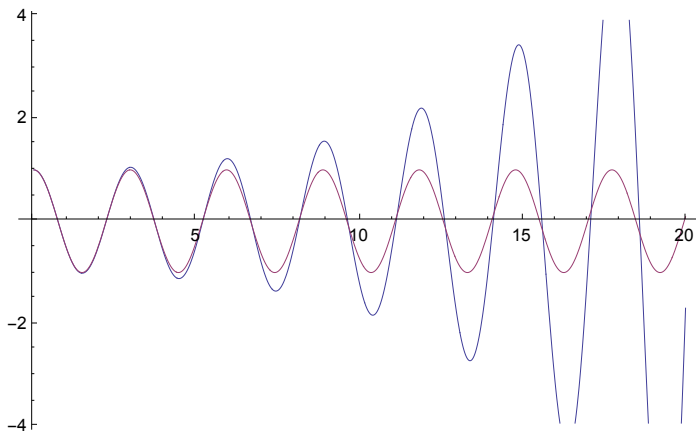
Figures 8

$$\mathbb{B}[x_-, \nu_-, b_-] := \frac{1}{\sqrt{1 + i \xi}} \text{Exp}\left[\frac{b^2 x \xi}{2(1 + \xi^2)}\right] \text{Exp}\left[i x \left(1 + \frac{b^2}{2(1 + \xi^2)}\right)\right] /. \xi \rightarrow \frac{x}{2\nu}$$

$\nu = 100;$

$b = 1.5;$

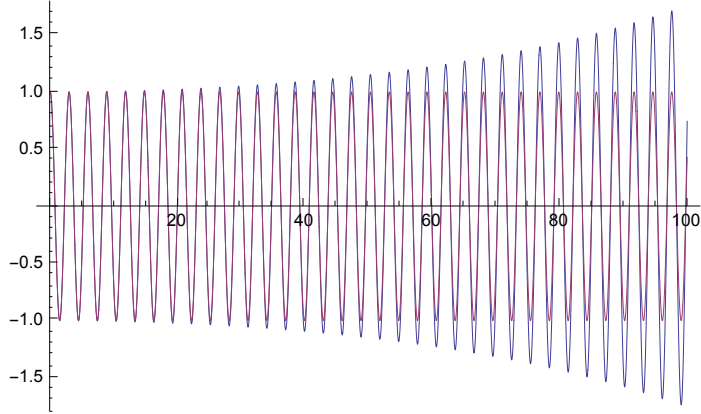
Plot[{{Re[B[x, ν, b]], Cos[x (1 + $\frac{b^2}{2}$)]}}, {x, 0, 20}]



$\nu = 10\,000;$

$b = 1.5;$

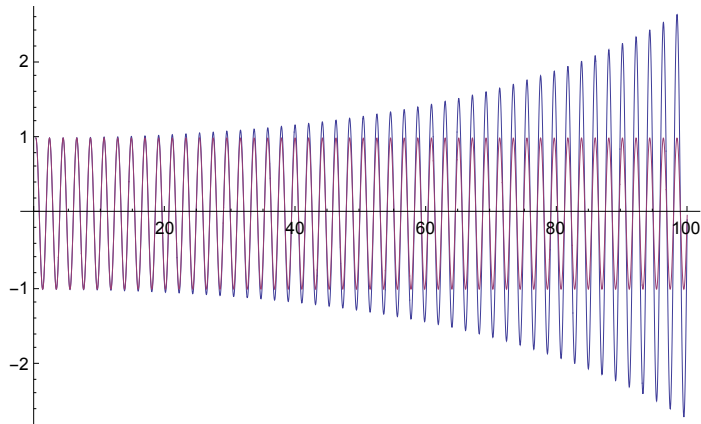
$\text{Plot}\left[\left\{\text{Re}[\mathbb{B}[x, \nu, b]], \text{Cos}\left[x\left(1 + \frac{b^2}{2}\right)\right]\right\}, \{x, 0, 100\}\right]$



$\nu = 10\,000;$

$b = 2;$

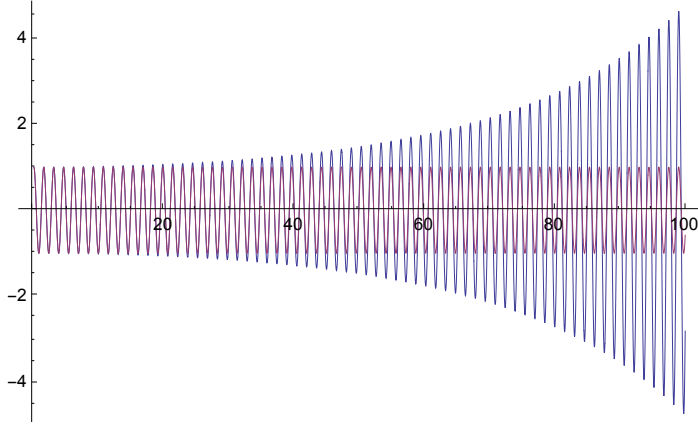
$\text{Plot}\left[\left\{\text{Re}[\mathbb{B}[x, \nu, b]], \text{Cos}\left[x\left(1 + \frac{b^2}{2}\right)\right]\right\}, \{x, 0, 100\}\right]$



```
v = 10 000;
```

```
b = 2.5;
```

```
Plot[{Re[B[x, v, b]], Cos[x (1 +  $\frac{b^2}{2}$ )]}, {x, 0, 100}]
```



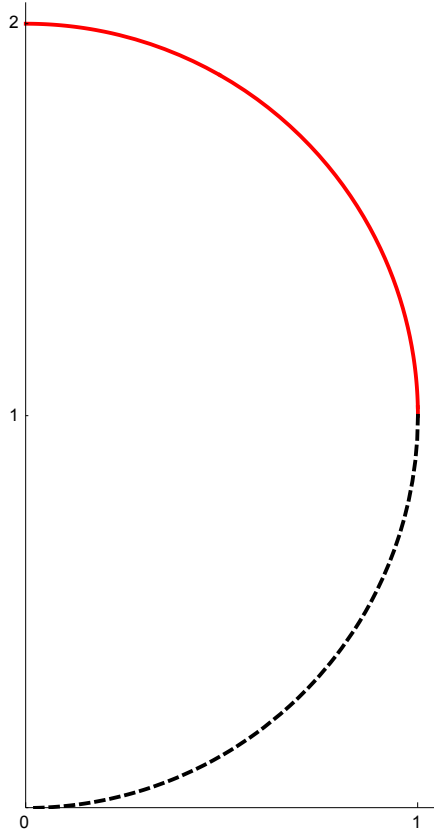
The final figure, though qualitatively similar to the others, cannot illustrate superoscillation because the assumption $b = 2.5 > b_{\max} = 2$ introduces terms with wavenumber > 1 into its construction.

Figure 9: Locus of saddlepoints, in the case $b = b_{\max}$

```

b = 2.0;
SemicircleSegment =
  ParametricPlot[{{ $\frac{b \xi}{1 + \xi^2}$ ,  $\frac{b}{1 + \xi^2}$ }, { $\xi$ , 0, 1}}, PlotRange -> {{0, 1.05}, {0, 2.05}},
  PlotStyle -> {Thick, Red}, Ticks -> {{0, 1}, {0, 1, 2}}];
SemicircleComplete = ParametricPlot[{{ $\frac{b \xi}{1 + \xi^2}$ ,  $\frac{b}{1 + \xi^2}$ },
  { $\xi$ , 1, 200}}, PlotRange -> {{0, 1.05}, {0, 2.05}},
  PlotStyle -> {Dashed, Thick, Black}, Ticks -> {{0, 1}, {0, 1, 2}}];
Show[SemicircleSegment, SemicircleComplete]

```



The red section is produced as ξ ranges from 0 to 1.

Figure 10: "Local wavenumber" \mathcal{K} in the case $b = b_{\max}$

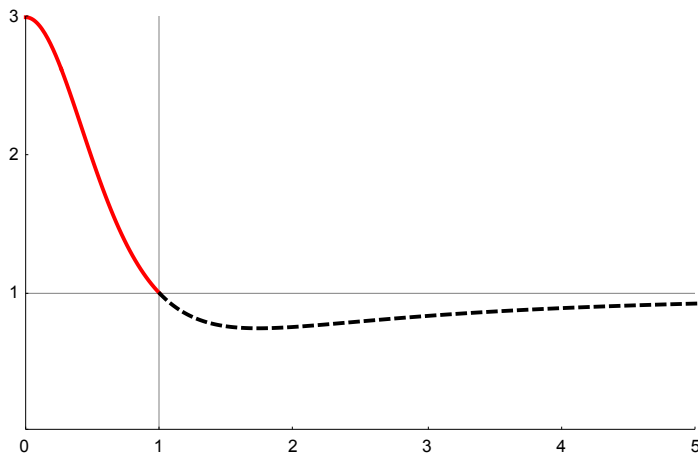
$$\mathcal{K}[b_-, \xi_-] := 1 + \frac{1}{2} b^2 \frac{1 - \xi^2}{(1 + \xi^2)^2}$$

LocalWavenumberSegment =

```
Plot[ $\mathcal{K}[2, \xi]$ , { $\xi$ , 0, 1}, PlotRange -> {{0, 5}, {0, 3}}, GridLines -> {{1}, {1}},  
PlotStyle -> {Thick, Red}, Ticks -> {{0, 1, 2, 3, 4, 5}, {0, 1, 2, 3}}];
```

```
LocalWavenumberComplete = Plot[ $\mathcal{K}[2, \xi]$ , { $\xi$ , 1, 5}, PlotRange -> {{0, 5}, {0, 3}},  
GridLines -> {{1}, {1}}, PlotStyle -> {Thick, Dashed, Black},  
Ticks -> {{0, 1, 2, 3, 4, 5}, {0, 1, 2, 3}}];
```

Show[LocalWavenumberSegment, LocalWavenumberComplete]



Only for $\xi < 1$ does \mathcal{K} have a superoscillatory value > 1 , and that value is bounded in the case b_{\max} by 3, as shown. The red portion of this curve can be identified with the red portion of Figure 9.

Figures 11

These are reconstructions of Berry's Figures 3a, 3b & 3c.

Figure 11a

$b = 2.0;$

$v = 12.5;$

Berry3a1 =

```
Plot[{Re[B[x, v, b]], Cos[3 x]}, {x, 0, 6.1}, PlotRange -> {{0, 6.2}, {-10, 5}}];
```

```
ExactDataPoints = Table[{n/2, Re[B[n/2, v, b] ε[n/2, v, b]]}, {n, 0, 12}];
```

```
Berry3a2 = ListPlot[ExactDataPoints, PlotStyle -> {PointSize[Large]}];
```

```
Show[Berry3a1, Berry3a2]
```

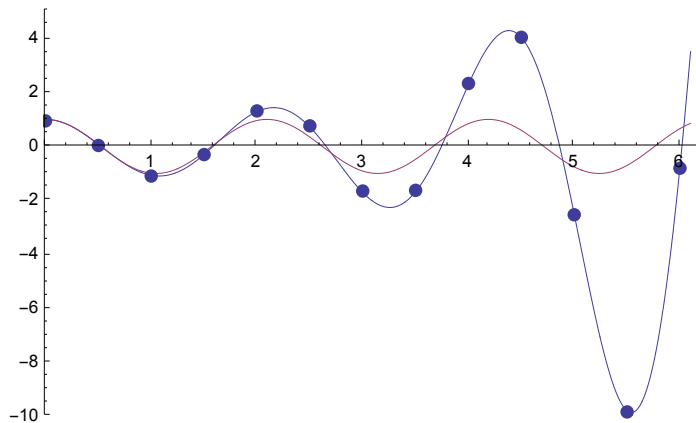


Figure 11aa

$b = 2.0;$

$v = 12.5;$

```
Berry3a1 = Plot[{Re[B[x, v, b]]}, {x, 0, 6},
```

```
PlotRange -> {{0, 6.2}, {-10, 5}}, PlotStyle -> {Black, Thick}];
```

```
Berry3aa1 = Plot[Re[B[x, v, b] ε[x, v, b]], {x, 0, 6},
```

```
PlotRange -> {{0, 6.2}, {-10, 5}}, PlotStyle -> {Red, Dashed, Thick}];
```

```
Show[Berry3a1, Berry3aa1]
```

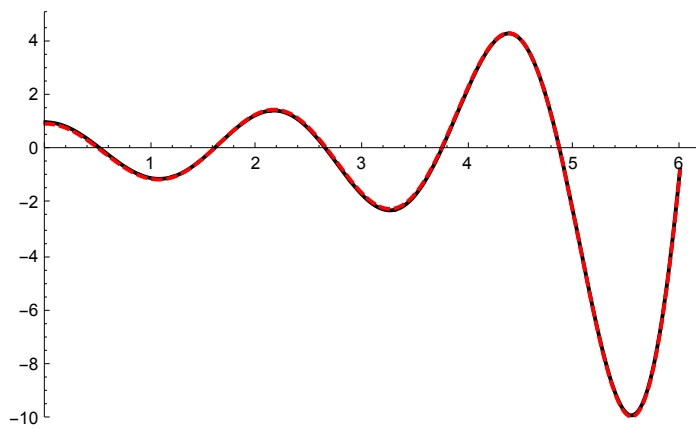


Figure 11b

```

b = 2.0;
v = 12.5;
Berry3b1 = Plot[{Re[B[x, v, b]]}, {x, 40, 46}];
ExactDataPoints3b = Table[{40 + n/2, Re[B[40 + n/2, v, b] ε[40 + n/2, v, b]]}, {n, 0, 12}];
Berry3b2 = ListPlot[ExactDataPoints3b, PlotStyle -> {PointSize[Large]}];
Show[Berry3b1, Berry3b2]

```

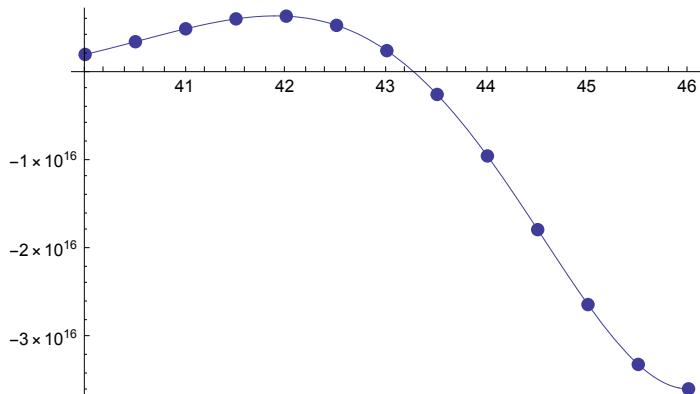


Figure 11c

```

b = 2.0;
v = 12.5;
Plot[Log[10, Abs[Re[B[x, v, b]]]], {x, 0, 30},
PlotRange -> {{0, 30}, {-4, 13}}, PlotStyle -> Thick,
GridLines -> {{0.55, 2.70, 21.25, 27.35}, {}},
GridLinesStyle -> Directive[Red]]

```

